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# بنية جبر هيكي للزمرة الخطية العامة

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# بنية جبر هيكي للزمرة الخطية العامة $GL(2, p^n)$


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**ملخص البحث**

يهـــدف هذا البحث إلى دراسة بنية جبر هيكي $E(N,\lambda)$ للزمره الخطية العامـــة مـــن المرتبة الثانية المرافقة للزوج $(N,\lambda)$ حيث $N$ هي زمرة المصفوفات التبديلـــية و $\lambda$ التمثيل الخطي للزمرة $N$ المعرف بدلالة تمثيل الإشارة. في هذا البحـــث يـــتم تحديـــد الأسـاس و معاملات البنية لهذا الجبر . كما نستخدم جبر الحدوديـــات لتحليل جبر هيكي عن طريق تحليل العنصر المحايد الى مجموع لعناصر جـــامدة ومـــتعامدة. أخيرا تستخدم تلك النتائج لتمييز العناصر الجامدة ولدراسة عناصر الوحدة في جبر هيكي.



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(1) If $\dim_k E(N,\lambda) \succ 2$ then evaluating $\det(A(x))$ with respect to the bottom row we see that $\det(A(x)) = 0$ , hence $x$ is not unit.

(2) If $\dim_k E(N,\lambda) = 2$ then , by 2.6, the characteristic of k must be 2. On the other hand $c_e = 0$ implies that $\det(A(x)) = 2c_0^2(q-1) = 0$ and so $x$ is not unit. □

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$e_1 = {}^{-1}\!/_{q+1}(a_0 - 2)$ and $e_2 = {}^{1}\!/_{q+1}(a_0 + (q-1))$ are two orthogonal idempotents in $E(N,\lambda)$ such that $1_{E(N,\lambda)} = e_1 + e_2$. □

Generally we may use the relations in 4.10 together with the action given in 4.11 to characterize the set of idempotents in the algebra $E(N,\lambda)$ as explained in the following proposition

**PROPOSITION 6.3** Suppose that $x = \sum_{\alpha\in X} c_\alpha a_\alpha \in E(N,\lambda)$. Then $x$ is idempotent if and only if the coefficients $c_\alpha$ satisfy the following identities

$$c_e(c_e - 1) + \sum_{e\neq\alpha\in X} 2(q-1)c_\alpha^2 = 0$$

$$c_0(2c_e - 1) - \sum_{e\neq\beta\in X} (q-3)c_\beta^2 = 0$$

$$c_\alpha(2c_e - 1) = 0 \ , \forall \alpha \in X \setminus \{0, e\}$$

**PROOF** Compare the coefficients in $x^2 = x$ and use the fact that $a_\alpha; \alpha \in X$ are linearly independent in $E(N,\lambda)$. □

Note that the identities in 6.3 hold in particular for the idempotents $e_1, e_2$ of $E(N,\lambda)$ which are defined in 6.1.

Next we consider the units ; that is the set of invertible elements ,of the Hecke algebra $E(N,\lambda)$. The following gives a partial characterization for the units in $E(N,\lambda)$.

**PROPOSITION 6.4** If $x = \sum_{\alpha\in X} c_\alpha a_\alpha \in E(N,\lambda)$ is a unit , then $c_e \neq 0$ .

**PROOF** Relative to the basis $\{a_\alpha; \alpha \in X\}$, the element $x = \sum_{\alpha\in X} c_\alpha a_\alpha \in E(N,\lambda)$ acts on $E(N,\lambda)$ according to the matrix

$$A(x) = \begin{array}{c} a_e \\ a_0 \\ a_\alpha \\ . \\ a_\beta \end{array} \begin{pmatrix} c_e & 2c_0(q-1) & 2c_\alpha(q-1) & . & 2c_\beta(q-1) \\ c_0 & c_e - c_0(q-3) & -c_\alpha(q-3) & . & -c_\beta(q-3) \\ c_\alpha & 0 & c_e & . & 0 \\ . & . & 0 & . & 0 \\ c_\beta & 0 & 0 & . & c_e \end{pmatrix}$$

Hence $x$ is a unit in $E(N,\lambda)$ if and only if A(x) is nonsingular ,that is if and only if $\det(A(x)) \neq 0$ . Now suppose that $c_e = 0$ then we consider two cases :

such that $A_1(X)\Phi_1(X)+A_2(X)\Phi_2(X)=1$. Let $e_i=A_i(a)\Phi_i(a); i=1,2$. Then both $E_1$ and $E_2$ are non-zero and $\mathbf{1}_A=e_1+e_2$ is an orthogonal idempotent decomposition in $A$ .

**PROOF** It is clear from the hypothesis that

$$e_1+e_2=A_1(a)\Phi_1(a)+A_2(a)\Phi_2(a)=1_A$$

Also , since $\Phi_1(a)\Phi_2(a)=\Phi(a)=0$, it follows that

$$e_1e_2=e_2e_1=A_1(a)A_2(a)\Phi_1(a)\Phi_2(a)=0$$

Therefore, $e_1=e_1\mathbf{1}_A=e_1(e_1+e_2)=e_1^2+e_1e_2=e_1^2$. Similarly $e_2^2=e_2$

Now to prove that $e_1\neq 0\neq e_2$ , suppose that $e_1=0$ ,then $A_1(a)\Phi_1(a)=0$ and so $\Phi(X)\mid A_1(X)\Phi_1(X)$. But this implies that $\Phi_2(X)$ divides both $A_1(X)\Phi_1(X)$ and $A_2(X)\Phi_2(X)$, hence

$$\Phi_2(X)\mid A_1(X)\Phi_1(X)+A_2(X)\Phi_2(X)=1$$

which is a contradiction and so $e_1\neq 0$. Similarly $e_2\neq 0$ . □

## §6. IDEMPOTENTS IN $E(N,\lambda)$ .

We shall apply the method described in the previous section to find an orthogonal idempotents decomposition of the identity of Hecke algebra $E(N,\lambda)$ . It is well known from the Brauer-Fitting theorem ( see [8], 1.4 ) that such decomposition of $a_e$ gives rise to a decomposition of $E(N,\lambda)$ into a direct sum of algebras and hence to a decomposition for the kG-module $Y(N,\lambda)$ .

**PROPOSITION 6.1** Suppose that $p\nmid q+1$, and let $e_1=-\frac{1}{q+1}(a_0-2)$, $e_2=\frac{1}{q+1}(a_0+(q-1))\in E(N,\lambda)$. Then $e_1,e_2$ are orthogonal idemoptents in $E(N,\lambda)$ such that $a_e(=1_{E(N,\lambda)})=e_1+e_2$.

**PROOF** Take $\alpha=0$ in 4.10. Then we have $a_0^2=2(q-1)-(q-3)a_0$, hence $a_0^2+(q-3)a_0-2(q-1)=0$ is the minimum equation of $a_0$, and so

6.2 $$X^2+(q-3)X-2(q-1)=(X-2)(X+(q-1))$$

is the minimum polynomial of $a_0$ with $A_1(X)=(X-2)$ and $A_2(X)=(X+(q-1))$ have no common divisor in k[X]. Now we may apply proposition 5.2 to the identity 6.2 and take $\Phi_1=-\frac{1}{q+1}$, $\Phi_2=\frac{1}{q+1}$ to get

**PROPOSITION 4.10** The Hecke algebra $E(N,\lambda)$ is generated as k-algebra by $\{ a_\alpha \ ; \alpha \in X \setminus \{e,0\}\}$ subject to the relations :

$$a_\alpha a_\beta = 0 \qquad \forall \alpha \neq \beta \ ,$$
$$a_\alpha^2 = 2(q-1)a_e - (q-3)a_0 \qquad \forall \alpha \in X \setminus \{e\}$$

□

From the above presentation of the Hecke algebra $E(N,\lambda)$ we deduce the following action of the elements of $E(N,\lambda)$ on the basis elements $a_\beta \ ; \beta \in X$

4.11 $$(\sum_{\alpha \in X} c_\alpha a_\alpha) a_\beta = 2c_\beta(q-1)a_e - c_\beta(q-3)a_0 + c_e a_\beta \quad \forall \ e \neq \beta \in X$$

## §5. POLYNOMIAL ALGEBRA AND IDEMPOTENTS

Constructing idempotents is an essential step towards analyzing any algebra. In this section we shall explain a technique for constructing idempotents in algebras by means of identities in the polynomial algebras. This method shall be applied in the next section to construct idempotents in the Hecke algebra $E(N,\lambda)$ considered in the previous sections. Suppose that $A$ is a finite dimensional k-algebra with an identity $1_A$ and that $a$ is an element of $A$ . Since A is finite dimensional ,there must be a positive integer $n$ such that the set $\{ 1, a, a^2, a^3, ..., a^n \}$ is linearly dependent ; if n is the least with this property then we have

5.1 $$\sum_{i=0}^{n} c_i a^i = 0 \quad \text{for some (not all zero)} \ c_i \in k$$

Equation 5.1 is called ***the minimum equation for a*** and the polynomial $\Phi(X) = \sum_{i=0}^{n} c_i X^i \in k[X]$ is called ***the minimum polynomial for a*** . If $\Omega(X)$ is any other polynomial in k[X] such that $\Omega(a) = 0$ then , using the minimality of $\Phi(X)$ , it is easy to see that $\Phi(X) \mid \Omega(X)$. The following shows how to construct , out of $\Phi(X)$ , an orthogonal idempotent decomposition of $1_A$ .

**PROPOSITION 5.2** Let A be a finite dimensional k-algebra and let $a \in A$ . Suppose that $\Phi(X) = \Phi_1(X)\Phi_2(X)$ is the minimum polynomial of $a$ , where $\Phi_1(X), \Phi_2(X)$ are non-constant polynomial in $k[X]$ with no common divisor in $k[X]$ . By Euclid's algorithm there exist $A_1, A_2 \in k[X]$

( In the later case $\alpha \neq \beta$, for if $\alpha = \beta$ then $b + \alpha^2 = 0$, but $b = -\beta = -\alpha$ and so $\alpha^2 = \alpha$, hence $\alpha = 1$.)

$\Leftrightarrow n_a^+ g_\beta n_b^+ g_\alpha = g_0 \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \vee g_0 \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$, hence giving coefficient

$\lambda(n_a^+)\lambda(n_b^+)\lambda\left(\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right) = 1$ in the first case, and the coefficient

$\lambda(n_a^+)\lambda(n_b^+)\lambda\left(\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}\right) = -1$. When we sum over all $a, b \in F_q^*$ we get $t_{\alpha,\beta,0} = 0$.

Now suppose that $0 \neq \gamma \in X$. Then

$$n_a^+ g_\beta n_b^+ g_\alpha \in g_\gamma N \Leftrightarrow [\, ab + a = b + \beta \;\wedge\; b + a\beta = \gamma(ab + a\alpha)]$$

$$\vee [\, b + a\beta = ab + a\alpha \;\wedge\; b + \beta = \gamma(ab + a)]$$

$$\Leftrightarrow [\, a = {}^{b+\beta}/_{b+1} \wedge b = {}^{\gamma\beta - \gamma a + \gamma a\alpha - a\beta}/_{1-\gamma}\,]$$

$$\vee [\, b = {}^{a\alpha - a\beta}/_{1-a} \wedge a = {}^{b+\beta}/_{\gamma(b+1)}]$$

Note that $b \neq -1$, $a \neq 1$; otherwise $\beta = 1$. The first case gives the coefficient $\lambda(n^+_{{}^{b+\beta}/_{b+1}})\lambda(n^+_{{}^{\gamma\beta-\gamma a+\gamma a\alpha-a\beta}/_{1-\gamma}})\lambda\left(\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right) = 1$, for all $b \neq -1$, $a \neq 1$. The second case contributes the coefficient $\lambda(n^+_{{}^{b+\beta}/_{\gamma(b+1)}})\lambda(n^+_{{}^{a(\alpha-\beta)}/_{1-a}})\lambda\left(\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}\right) = -1$, for all $b \neq -1$, $a \neq 1$. Clearly these scalars cancel each other when we sum over all such $a$ and $b$. This proves that $t_{\alpha,\beta,\gamma} = 0$.

Summarizing we have the following

**THEOREM 4.9** The structure constants $t_{\alpha,\beta,\gamma}$ of the Hecke algebra $E(N,\lambda)$ are given as follows :

$$t_{\alpha,\beta,\gamma} = \begin{cases} -(q-3) & \text{if } \alpha = \beta = \gamma = 0 \\ 2(q-1) & \text{if } \alpha = \beta \text{ and } \gamma = e \\ 0 & \text{otherwise} \end{cases}$$ □

Now we translate theorem 4.9 into the following presentation for the Hecke algebra $E(N,\lambda)$

$(a=\gamma^{-1}\beta \wedge b={}^{\beta-\beta\gamma^{-1}}\!/_{\beta\gamma^{-1}-1}) \vee (a=\beta \;\; \wedge b={}^{\beta\gamma-\beta}\!/_{1-\beta\gamma}))$ ( note that $\gamma\beta \neq 1$ by our choice of the index set $X \subseteq F_q$. ). In the first case we have $n_a^+ g_\beta n_b^+ g_0 = \begin{pmatrix} 1 & 1 \\ 1 & \gamma \end{pmatrix}\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ which gives the coefficient +1, and in the second case we get $n_a^+ g_\beta n_b^+ g_0 = \begin{pmatrix} 1 & 1 \\ 1 & \gamma \end{pmatrix}\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ which gives the coefficient -1 , hence we get 0 when we sum over all a,b. This proves the following

**PROPOSITION 4.7** If $\beta \neq 0$ ,then $t_{0,\beta,\gamma} = 0$ for all $\gamma \in X$. □

**(3) *The case when*** $\alpha \neq 0 \neq \beta$

In this case , $sg_\beta v g_\alpha ; s \in R_\beta, v \in R_\alpha$, takes the following form :

$$(*) \qquad n_a^+ g_\beta n_b^+ g_\alpha = \begin{pmatrix} a(1+b) & ab+a\alpha \\ b+\beta & b+\alpha\beta \end{pmatrix}$$

Now $(*) \in N \Leftrightarrow (b=-1 \wedge \alpha\beta=1) \vee (\alpha=\beta=-b)$ , the first case is rejected since $\alpha\beta \neq 1$, by our choice of the index set $X$. The second case is valid if and only if $b=\alpha$ which gives the set

$$\left\{ n_a^+ g_\beta n_{-\alpha^{-1}}^+ g_\alpha = \begin{pmatrix} a(1-\alpha) & 0 \\ 0 & \alpha+\beta \end{pmatrix} ; a \in F_q^* \right\}$$

, each member of this set contributes the coefficient $\lambda(n_a^+)\lambda(n_{-\alpha^{-1}}^+)\lambda(\begin{pmatrix} a(1-\alpha) & 0 \\ 0 & \alpha+\beta \end{pmatrix}) = 1$. Hence when we sum over all $a \in F_q^*$ we get q-1. This proves the following

**PROPOSITION 4.8** Suppose that $\alpha, \beta \in X$ with $\alpha \neq 0 \neq \beta$ . Then

(1) $t_{\alpha,\beta,e} \neq 0 \;\; \Leftrightarrow \;\; \alpha=\beta$ ,

(2) $t_{\alpha,\alpha,e} = q-1$ ,

(3) $t_{\alpha,\beta,\gamma} = 0$ for all $\gamma \; (\neq e) \in X$. □

Also,

$(*) \in g_0 N \Leftrightarrow [b+\alpha\beta=0 \wedge a(b+1)=b+\beta\;] \vee [b+\beta=0 \wedge a(b+\alpha)=b+\alpha\beta]$

$\Leftrightarrow [b=-\alpha\beta \wedge a={}^{\beta-\alpha\beta}\!/_{1-\alpha\beta}] \vee [\,b=-\beta \wedge a={}^{\alpha\beta-\beta}\!/_{\alpha-\beta}]$

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & \beta \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a(b+1) & ab \\ b+\beta & b \end{pmatrix} \quad \text{or}$$

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & \beta \end{pmatrix}\begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (ab+1) & a \\ b+\beta & \beta \end{pmatrix}$$

Since $b, ab, a, \beta \neq 0$, it follows that $sg_\beta v g_0 \notin N$, for all $s \in R_\beta, v \in R_0$. Therefore we have $t_{0,\beta,e} = 0$. On the other hand $\begin{pmatrix} a(b+1) & ab \\ b+\beta & b \end{pmatrix} \in g_0 N = \left\{ \begin{pmatrix} x & y \\ x & 0 \end{pmatrix}, \begin{pmatrix} y & x \\ 0 & x \end{pmatrix}; x, y \in F_q^* \right\}$ if and only if $b+\beta = 0 \wedge ab = b$, that is if and only if $b = -\beta \wedge a = 1$, in which case $\begin{pmatrix} a(b+1) & ab \\ b+\beta & b \end{pmatrix} = \begin{pmatrix} b+1 & b \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ b+1 & 0 \end{pmatrix}$, hence contributes the coefficient

4.5 $$\lambda(n_1^+)\lambda(n_{-\beta}^+)\lambda(\begin{pmatrix} 0 & -\beta \\ 1-\beta & 0 \end{pmatrix}) = -1$$

While $\begin{pmatrix} a(b+1) & a \\ b+\beta & \beta \end{pmatrix} \in g_0 N = \left\{ \begin{pmatrix} x & y \\ x & 0 \end{pmatrix}, \begin{pmatrix} y & x \\ 0 & x \end{pmatrix}; x, y \in F_q^* \right\}$ if and only if $b+\beta = 0 \wedge a = \beta$, that is if and only if $b = -\beta \wedge a = \beta$ in which case $\begin{pmatrix} a(b+1) & a \\ b+\beta & \beta \end{pmatrix} = \begin{pmatrix} -\beta^2+\beta & \beta \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \beta \\ -\beta^2+\beta & 0 \end{pmatrix}$, hence gives the coefficient

4.6 $$\lambda(n_a^+)\lambda(n_b^-)\lambda(\begin{pmatrix} 0 & \beta \\ -\beta^2+\beta & 0 \end{pmatrix}) = 1$$

By summing 4.5 and 4.6 we get $t_{0,\beta,0} = 0$. Now if $\gamma \in X \setminus \{0,1\}$, then

$$\begin{pmatrix} a(b+1) & ab \\ b+\beta & b \end{pmatrix} \in g_\gamma N = \left\{ \begin{pmatrix} x & y \\ x & \gamma y \end{pmatrix}, \begin{pmatrix} y & x \\ \gamma y & x \end{pmatrix}; x, y \in F_q^* \right\} \Leftrightarrow$$

$$\Leftrightarrow (a(b+1) = b+\beta \wedge \gamma ab = b) \vee (ab = b \wedge b+\beta = \gamma a(b+1))$$

$$\Leftrightarrow \quad (b = {}^{1-\gamma^{-1}}\!/_{\gamma^{-1}-\beta} \wedge a = \gamma^{-1}) \vee (a = 1 \wedge b = {}^{\gamma-\beta}\!/_{1-\gamma}) \quad (\text{ note that } \gamma \neq 1 \text{ })$$

$$\Leftrightarrow n_a^+ g_\beta n_b^+ g_0 = \begin{pmatrix} 1 & 1 \\ 1 & \gamma \end{pmatrix}\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \quad \vee \quad n_a^+ g_\beta n_b^+ g_0 = \begin{pmatrix} 1 & 1 \\ 1 & \gamma \end{pmatrix}\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \text{ which}$$

contributes the coefficients +1 , -1 , respectively cancelling each other when we sum over a,b. similarly

$$\begin{pmatrix} a(b+1) & a \\ b+\beta & \beta \end{pmatrix} \in g_\gamma N = \left\{ \begin{pmatrix} x & y \\ x & \gamma y \end{pmatrix}, \begin{pmatrix} y & x \\ \gamma y & x \end{pmatrix}; x, y \in F_q^* \right\} \Leftrightarrow$$

Also $n_a^- g_0 n_b^+ g_0 \in g_0 N \Leftrightarrow b+1=0, ab=b \Leftrightarrow b=-1, a=1$ , in which case $n_1^- g_0 n_{-1}^+ g_0 = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ which contributes the coefficient $\lambda(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})\lambda(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix})\lambda(\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}) = 1$ . Summing up the coefficients from both cases we get

**PROPOSITION 4.3** $\quad t_{0,0,0} = -(q-3)$. □

Now for an arbitrary $\gamma \in F_q^*$ we have

$g_\gamma N = \left\{ \begin{pmatrix} x & y \\ x & \gamma y \end{pmatrix}, \begin{pmatrix} y & x \\ \gamma y & x \end{pmatrix}; x, y \in F_q^* \right\}$ . Therefore $n_a^+ g_0 n_b^- g_0$, $n_a^- g_0 n_b^- g_0 \notin g_\gamma N$ for all $a, b \in F_q^*$, while $n_a^+ g_0 n_b^+ g_0 \in g_\gamma N$ if and only if $[ab + a = b \wedge b = \gamma ab] \vee [ab = b \wedge b = \gamma(ab + a]$

$$\Leftrightarrow [a = \frac{b}{b+1} \wedge \gamma = \frac{b+1}{b}] \vee [a = 1 \wedge \gamma = \frac{b}{b+1}] \ ; \ b \neq 0, -1$$

in which case $n_{b/_{b+1}}^+ g_0 n_b^+ g_0 = \begin{pmatrix} 1 & 1 \\ 1 & {}^{b+1}/_b \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & {}^{b^2}/_{b+1} \end{pmatrix}$

and $\qquad n_1^+ g_0 n_b^+ g_0 = \begin{pmatrix} 1 & 1 \\ 1 & {}^{b}/_{b+1} \end{pmatrix} \begin{pmatrix} 0 & {}^{(b+1)^2}/_b \\ b+1 & 0 \end{pmatrix}$

contributing coefficients 1 and -1 , respectively, hence canceling each other when summing over all $b$. Similarly

$n_a^- g_0 n_b^+ g_0 \in g_\gamma N \Leftrightarrow [ab = b+1 \wedge b = \gamma(b+1)] \vee [ab = b \wedge b+1 = \gamma b]$

$\Leftrightarrow [a = {}^{b+1}/_b \wedge \gamma = {}^{b}/_{b+1}] \vee [a = 1 \wedge \gamma = {}^{b+1}/_b]$ , in which case

$n_{b+1/_b}^- g_0 n_b^+ g_0 = \begin{pmatrix} 1 & 1 \\ 1 & \gamma \end{pmatrix} \begin{pmatrix} b+1 & 0 \\ 0 & b+1 \end{pmatrix} \vee n_b^- g_0 n_b^+ g_0 = \begin{pmatrix} 1 & 1 \\ 1 & \gamma \end{pmatrix} \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$,

contributing coefficients -1 , 1 ; respectively , hence canceling each other when summing over all b. This proves the following

**PROPOSITION 4.4** $\quad t_{0,0,\gamma} = 0$ for all $\gamma (\neq e, 0) \in X$ . □

**(2) *The case when*** $\alpha = 0$ ***and*** $\beta \neq 0$**:**

Consider the elements $s g_\beta v g_0$, where $s \in R_\beta$ and $v \in R_0$. By our choice of those transversal, $s g_\beta v g_0$ takes one of the following two forms

$$n_a^+ g_0 n_b^+ g_0 = \begin{pmatrix} ab+a & ab \\ b & b \end{pmatrix}$$

$$n_a^+ g_0 n_b^- g_0 = \begin{pmatrix} ab+a & a \\ b & 0 \end{pmatrix}$$

$$n_a^- g_0 n_b^- g_0 = \begin{pmatrix} ab & 0 \\ b+1 & 1 \end{pmatrix}$$

$$n_a^- g_0 n_b^+ g_0 = \begin{pmatrix} ab & ab \\ b+1 & b \end{pmatrix}$$

Now for all $a,b \in F_q^*$, we have $n_a^+ g_0 n_b^+ g_0$ , $n_a^- g_0 n_b^+ g_0 \notin N$. On the other hand $n_a^+ g_0 n_b^- g_0 \in N$ if and only if $b=-1$ in which case $n_a^+ g_0 n_b^- g_0 = \begin{pmatrix} 0 & a \\ -1 & 0 \end{pmatrix}$. This contributes the coefficient $\lambda(n_a^+)\lambda(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})\lambda(\begin{pmatrix} 0 & a \\ -1 & 0 \end{pmatrix}) = 1$ for all $a \in F_q^*$ giving a total coefficient equals (q-1). Similarly $n_a^- g_0 n_b^- g_0 \in N$ if and only if $b=-1$ in which case $n_a^- g_0 n_b^- g_0 = \begin{pmatrix} -1 & 0 \\ 0 & a \end{pmatrix}$. This contributes the coefficient $\lambda(n_a^-)\lambda(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})\lambda(\begin{pmatrix} -1 & 0 \\ 0 & a \end{pmatrix}) = 1$ , giving (q-1) as a total coefficient. This proves the following

**PROPOSITION 4.2** $\quad t_{0,0,e} = 2(q-1)$. □

To evaluate the constant $t_{0,0,0}$ , we replace N by $g_0 N$ in the above discussion and note that $g_0 N = \left\{ \begin{pmatrix} x & y \\ x & 0 \end{pmatrix}, \begin{pmatrix} y & x \\ 0 & x \end{pmatrix} ; x,y \in F_q^* \right\}$ . Therefore $n_a^+ g_0 n_b^+ g_0, n_a^- g_0 n_b^- g_0 \notin g_0 N$. On the other hand $n_a^+ g_0 n_b^- g_0 \in g_0 N$ if and only if $ab+a=b$ in which case $n_a^+ g_0 n_b^- g_0 = \begin{pmatrix} b & b/_{b+1} \\ b & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & b/_{b+1} \end{pmatrix}$ which contributes the following coefficient

$$\sum_{-1 \neq b \in F_q^*} [\lambda(n^+_{b/_{b+1}})\lambda(n_b^-)\lambda(\begin{pmatrix} b & 0 \\ 0 & b/_{b+1} \end{pmatrix}) = -1\ ] = -(q-2).$$

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}(N \cap {}^{g_\alpha}N) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} y^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix}(N \cap {}^{g_\alpha}N) = \begin{pmatrix} xy^{-1} & 0 \\ 0 & 1 \end{pmatrix}(N \cap {}^{g_\alpha}N),$$

and

$$\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}(N \cap {}^{g_\alpha}N) = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\begin{pmatrix} 0 & y^{-1} \\ \alpha^{-1}y^{-1} & 0 \end{pmatrix}(N \cap {}^{g_\alpha}N) = \begin{pmatrix} \alpha^{-1}xy^{-1} & 0 \\ 0 & 1 \end{pmatrix}(N \cap {}^{g_\alpha}N)$$

, we may take $R_\alpha = \left\{ n_a^+ = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}; a \in F_q^* \right\}$.

## §4 THE STRUCTURE CONSTANTS OF $E(N,\lambda)$

For every $\alpha \in F_q^* \setminus \{1,-1\}$, let $a_\alpha \in E(N,\lambda)$ be the operator given by $a_\alpha([N_\lambda]) = \sum_{r\in R_\alpha} \lambda(r) r g_\alpha [N]_\lambda$ . Then from the results of §2 we saw that $E(N,\lambda) = k - a\lg\text{ebra}\langle a_e, a_\alpha ; \alpha \in F_q^* \setminus \{1,-1\}\rangle$, where $a_e$ is the identity operator of $E(N,\lambda)$;that is the operator which corresponds to $N$. Also for all $\alpha,\beta \in F_q \setminus \{1,-1\}$ we have $a_\alpha a_\beta = \sum_{\gamma\in X} t_{\alpha,\beta,\gamma}\, a_\gamma$ , where $X$ is a subset of $F_q$ which indexes the basis elements of $E(N,\lambda)$ . Since $D_\alpha = D_{\alpha^{-1}}$ for all $\alpha \in F_q^*$, it follows that $a_\alpha = a_{\alpha^{-1}}$ and so we may choose the index set $X \subseteq F_q^*$ so that $\alpha\beta \neq 1$ for all $\alpha,\beta \in X$. We also take $0, e \in X$ . From proposition 1.2 we have

4.1 $$t_{\alpha,\beta,\gamma} = \sum_{\substack{r\in R_\beta, v\in R_\alpha \\ r g_\beta v g_\alpha = g_\gamma n \in g_\gamma N}} \lambda(r)\lambda(v)\lambda(n)$$

To determine $t_{\alpha,\beta,\gamma}$ and since $R_\alpha$ depends on weather $\alpha = 0$ *or* $\alpha \neq 0$ and because $E(N,\lambda)$ is commutative we only need to consider the following three cases :

(1) $\alpha = \beta = 0$ , (2) $\alpha = 0$ *and* $\beta \neq 0$ , (3) $\alpha \neq 0 \neq \beta$

(1)***The case when*** $\alpha = \beta = 0$ : For all $a, b \in F_q^*$ we have

Hence we have the following formulae which determines the dimension of the Hecke algebra $E(N,\lambda)$.

**THEOREM 2.6** $\dim_k E(N,\lambda) = \begin{cases} \frac{1}{2}(q-3)+2 & \text{if } p \neq 2 \\ \frac{1}{2}(q-2)+2 & \text{if } p = 2 \end{cases}$ □

## §3. THE TRANSVERSAL $R_\alpha$

In order to find the structure constants for the algebra $E(N,\lambda)$ using the method described in §1, we need to choose a suitable transversal $R_\alpha$ for $\{x(N \cap {}^{g_\alpha}N);\, x \in N\}$, $\alpha \in F_q^* \setminus \{1,-1\}$. From 2.4 and 2.5 we need to distinguish two cases :

(1)***The case when*** $\alpha = 0$. We have $N \cap {}^{g_0}N = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}; x \in F_q^* \right\}$. Hence

$$\left|N \cap {}^{g_0}N\right| = q-1 \text{ and so } |R_0| = \frac{2(q-1)^2}{q-1} = 2(q-1). \text{ Since}$$

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}(N \cap {}^{g_0}N) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} y^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix}(N \cap {}^{g_0}N) = \begin{pmatrix} xy^{-1} & 0 \\ 0 & 1 \end{pmatrix}(N \cap {}^{g_0}N),$$

and

$$\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}(N \cap {}^{g_0}N) = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\begin{pmatrix} y^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix}(N \cap {}^{g_0}N) = \begin{pmatrix} 0 & xy^{-1} \\ 1 & 0 \end{pmatrix}(N \cap {}^{g_0}N), \text{ we}$$

may take $R_0 = \left\{ n_a^+ = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, n_a^- = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}; a \in F_q^* \right\}$.

(2)***The case when*** $\alpha \in F_q^* \setminus \{1,-1\}$. By 2.4(2) we have

$N \cap {}^{g_\alpha}N = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} 0 & \alpha x \\ x & 0 \end{pmatrix}; x \in F_q^* \right\}$ and so $\left|N \cap {}^{g_\alpha}N\right| = 2(q-1)$, hence

$|R_\alpha| = \dfrac{2(q-1)^2}{2(q-1)} = q-1$. Since

$$g_\alpha^{-1}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} g_\alpha = \begin{pmatrix} \dfrac{\alpha x - y}{\alpha - 1} & \dfrac{\alpha(x-y)}{\alpha - 1} \\ \\ \dfrac{y-x}{\alpha - 1} & \dfrac{\alpha y - x}{\alpha - 1} \end{pmatrix}$$

and , $g_\alpha^{-1}\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} g_\alpha = \begin{pmatrix} \dfrac{\alpha x - y}{\alpha - 1} & \dfrac{\alpha^2 x - y}{\alpha - 1} \\ \\ \dfrac{y-x}{\alpha - 1} & \dfrac{y - \alpha x}{\alpha - 1} \end{pmatrix}$. Therefore

$g_\alpha^{-1}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} g_\alpha \in N \Leftrightarrow \textit{either}\ (\alpha = -1, x = -y)\ \textit{or}\ (x = y)$. Similarly

$g_\alpha^{-1}\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} g_\alpha \in N \Leftrightarrow \textit{either}\ (\alpha = -1, x = y)\ \textit{or}\ (\alpha x = y)$.

The following determines the $\lambda$-compatible $(N,N)$-cosets of GL(2,q)

**PROPOSITION 2.5** The coset $D_\alpha = Ng_\alpha N$ is $\lambda$-compatible if and only if $\alpha \neq -1$.

**PROOF** If $\alpha \neq -1$, then from 2.4 each element of $N^{g_\alpha} \cap N$ is either of the form $g_\alpha^{-1}\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} g_\alpha \in N_1$ or $g_\alpha^{-1}\begin{pmatrix} 0 & x \\ \alpha x & 0 \end{pmatrix} g_\alpha \in N_2$. Therefore $\lambda^{g_\alpha}\Big|_{N^{g_\alpha} \cap N} = \lambda$ and so $D_\alpha$ is $\lambda$-compatible. Conversely suppose that $\alpha = -1$ then ,again by 2.4 , each element of $N^{g_{-1}} \cap N$ is of the form $g_{-1}^{-1}\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} g_{-1} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. But $\lambda^{g_{-1}}(g_{-1}^{-1}\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} g_{-1}) = \lambda(\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}) = -1$, while $\lambda(g_{-1}^{-1}\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} g_{-1}) = \lambda(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}) = 1$. Hence $D_{-1}$ is not $\lambda$-compatible. □

If we denote by $\langle N \backslash G / N \rangle_\lambda$ the set of $\lambda$-compatible $(N,N)$-cosets in $GL(2,q)$ then from the previous results we conclude that
$\langle N \backslash G / N \rangle_\lambda = \{N, D_0 = Ng_0N, D_\alpha = Ng_\alpha N\ (= D_{\alpha^{-1}});\ \alpha \in F_q^* \setminus \{1,-1\}\}$

The following table gives the size of each $(N,N)$-double cosets in GL(2,q).

| D | N | $D_0$ | $D_{-1}$ | $D_\alpha(=D_{\alpha^{-1}});\alpha\neq 0,1,-1$ |
|---|---|---|---|---|
| $\lvert D\rvert$ | $2(q-1)^2$ | $4(q-1)^3$ | $(q-1)^3$ | $2(q-1)^3$ |

Note that
$2(q-1)^2+4(q-1)^3+(q-1)^3+(q-3)(q-1)^3=(q-1)^2(q^2+q)=|GL(2,q)|$

In order to determine the dimension of the Hecke algebra $E(N,\lambda)$ we need to determine the $\lambda$-compatible $(N,N)$-cosets. For that purpose we first describe the subgroups $N\cap^{g_\alpha}N;\ \alpha\in F_q-\{1\}$.

**PROPOSITION 2.4** For each $\alpha\in F_q-\{1\}$, we have

(1) If $\alpha=0$, then $N\cap^{g_\alpha}N=\left\{\begin{pmatrix}x&0\\0&x\end{pmatrix};x\in F_q^*\right\}$

(2) *If* $\alpha=-1$ , *then* $N\cap^{g_\alpha}N=$

$\left\{\ g_{-1}^{-1}\begin{pmatrix}x&0\\0&-x\end{pmatrix}g_{-1}\ ,g_{-1}^{-1}\begin{pmatrix}x&0\\0&x\end{pmatrix}g_{-1},\ g_{-1}^{-1}\begin{pmatrix}0&x\\-x&0\end{pmatrix}g_{-1},\ g_{-1}^{-1}\begin{pmatrix}0&x\\x&0\end{pmatrix}g_{-1};x\in F_q^*\right\}$

$N\cap^{g_\alpha}N=\left\{g_\alpha^{-1}\begin{pmatrix}x&0\\0&x\end{pmatrix}g_\alpha\ ,g_\alpha^{-1}\begin{pmatrix}0&x\\\alpha x&0\end{pmatrix}g_\alpha\,;\,x\in F_q^*\right\}$ when $\alpha\neq 0,-1$.

**PROOF**

(1) If $x,y\in F_q^*$, then $g_0^{-1}\begin{pmatrix}x&0\\0&y\end{pmatrix}g_0=\begin{pmatrix}y&0\\x-y&x\end{pmatrix}\in N\Leftrightarrow x=y$. On the other hand $g_0^{-1}\begin{pmatrix}0&x\\y&0\end{pmatrix}g_0=\begin{pmatrix}y&y\\x-y&-y\end{pmatrix}\notin N$ . Therefore we have

$N\cap^{g_0}N=\left\{\begin{pmatrix}x&0\\0&x\end{pmatrix};x\in F_q^*\right\}$.

(2) Now suppose that $\alpha\in F_q^*-\{1\}$. Then

**PROOF** Suppose that $g \in GL(2,q)$. Then the number of entries of g which equal 0 is either (1) two, (2) one or (3) none. In the first case $g \in N$ and so $NgN = N$.

(2) If $g = \begin{pmatrix} 0 & x \\ y & z \end{pmatrix}; x,y,z \in \mathbf{F}_q^*$, then $\begin{pmatrix} z & y \\ x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in NgN$

and so $NgN = D_0$. The same is true if $g = \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}$ or $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$.

(3) If $g = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$; $x, y, z, \in \mathbf{F}_q^* - \{1\}$ and $t \in \mathbf{F}_q - \{1\}$, then

$$\begin{pmatrix} 1 & 1 \\ 1 & y^{-1}z^{-1}xt \end{pmatrix} = \begin{pmatrix} x^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ 0 & xy^{-1} \end{pmatrix} \in NgN$$

Therefore $NgN = D_\alpha$; where $\alpha = y^{-1}z^{-1}xt$. □

**DEFINITION** If $g = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in GL(2,q)$, define $\pi(g) = y^{-1}z^{-1}xt$.

**LEMMA 2.2** If $g' = n_1 g n_2 \in NgN$ then $\pi(g) = \pi(g')$ or $\pi(g) = \pi(g')^{-1}$. Hence $D_\alpha = D_{\alpha^{-1}}$ for all $\alpha \in \mathbf{F}_q^* - \{1\}$.

**PROOF** : If $n_1, n_2 \in N_1$ then $\pi(g) = \pi(g')$. If $n_1 \in N_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} N_1$ and $n_2 \in N_1$ then $\pi(g') = \pi(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x) = \pi(x)^{-1} = \pi(g)^{-1}$, where $x \in NgN$.

Similarly if $n_1 \in N_1, n_2 \in N_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} N_1$, then $\pi(g') = \pi(g)^{-1}$ and if $n_1, n_2$ are either in $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} N_1$ or $N_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $\pi(g) = \pi(g')$. □

The above lemma implies at once the following

**PROPOSITION 2.3** (1) The Hecke algebra $E(N, \lambda)$ is commutative.
(2) If $p = 0$, then the kG-module $Y(N, \lambda)$ is multiplicity free.

**PROOF** (1) This follows from ( [7], p.28 ), since $D_\alpha = D_{\alpha^{-1}}$ from lemma 2.2.
(2) See ([4], p.306 Exercise 18 ). □

**REMARK** When $\lambda = 1_H$; the trivial character of $H$ , then the formula in 1.2 coincides with the one proved in ( [7], p.15 ).

Now we take $G=GL(n,q)$ ; the general linear group defined over the finite field $\mathbf{F_q}$, where **q** is a power of prime number $p$ and let $N$ be the set of all monomial ( permutation ) matrices in $G$. Then $N$ is a subgroup of G in which every matrix has a unique non-zero coefficient in each row and in each column . Hence there is a group epimorphism $\mu : N \to S_n$ with $\ker \mu = T$; the set of diagonal matrices in $G$. Therefore we may lift the sign representation $\varepsilon$ of $S_n$ to a representation $\lambda$ of $N$ via $\mu$; that is we let $\lambda(n) := \varepsilon(\mu(n))$ for all $n \in N$. We are interested in the Hecke algebra $E(N,\lambda)$ and we shall concentrate on the case when n=2.

## §2. THE CASE WHEN $G=GL(2,q)$

We now take $G=GL(2,q)$, where $q$ is a power of a prime number $p$. Then $N = N_1 \cup N_2$; where $N_1 = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}; \ x, y \in F_q, xy \neq 0 \right\}$ and $N_2 = \left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}; x, y \in F_q, xy \neq 0 \right\}$ . The multiplicative character $\lambda : N \to k^\times$ defined above is then given by

$$\lambda(n) = \begin{cases} 1 & \text{if } n \in N_1 \\ -1 & \text{if } n \in N_2 \end{cases}$$

Write $p$ for the characteristic of k and note that $\lambda = 1_N$ if $p = 2$.

In order to study the structure of the Hecke algebra $E(N,\lambda)$, the first step is to find a k-basis for this algebra . As we have seen in the previous section this is equivalent to determining the set of $\lambda$-compatible $(N,N)$-cosets in $G$. First we shall determine a transversal for the set of double cosets of the subgroup N in GL(2,q) . For each $\alpha \in \mathbf{F_q} - \{1\}$, let $g_\alpha = \begin{pmatrix} 1 & 1 \\ 1 & \alpha \end{pmatrix} \in GL(2,q)$.

**PROPOSITION 2.1** $\{ N,\ D_\alpha = N g_\alpha N; \alpha \in \mathbf{F_q} - \{1\} \}$ is the set of (N,N)-cosets in $GL(2,q)$.

**LEMMA 1.1** ( [2], 2.2)

(1) $\dim_k(E(H,\lambda)) = |J_\lambda|$.

(2) $\{a_x, x \in J_\lambda\}$ is a k-basis of $E(H,\lambda)$. If $a_x a_y = \sum t_{x,y,z} a_z$, $(x,y,z \in J_\lambda)$, then $t_{x,y,z}$ all belong to the subring of k generated by $\lambda(H)$. □

**DEFINITION** If $x \in J_\lambda$, then the coset $NxN$ is called $\lambda$*-compatible.*

The k-algebra $E(H,\lambda)$ is called the ***Hecke algebra*** associated with the triple $(G,H,\lambda)$. The scalars $t_{x,y,z}$ are called the ***structure constants*** of the algebra $E(H,\lambda)$. It is known ( see [9],§1.5 ) that any finite dimensional algebra is determined ( up to isomorphism ) by its structure constants. To determine the constants $t_{x,y,z}$ , we note that

$$\begin{aligned} a_x a_y([H]_\lambda) &= \sum_{z \in J_\lambda} t_{x,y,z} a_z([H]_\lambda) \\ &= \sum_{z \in J_\lambda} t_{x,y,z} \sum_{s \in R_z} \lambda(s) sz[H]_\lambda \\ &= \sum_{z \in J_\lambda} \sum_{s \in R_z} t_{x,y,z} \lambda(s) sz[H]_\lambda \\ &= \sum_{z \in J_\lambda} (t_{x,y,z} z[H]_\lambda + \sum_{1 \neq s \in R_z} t_{x,y,z} \lambda(s) sz[H]_\lambda) \end{aligned}$$

Therefore $t_{x,y,z} = \textit{coeffecient of } z[H]_\lambda \textit{ in } a_x a_y([H]_\lambda)$. On the other hand

$$\begin{aligned} a_x a_y([H]_\lambda) &= a_x(\sum_{z \in R_y} \lambda(r) ry[H]_\lambda) \\ &= \sum_{r \in R_y} \lambda(r) ry a_x([H]_\lambda) \\ &= \sum_{r \in R_y} \lambda(r) ry \sum_{v \in R_x} \lambda(v) vx[H]_\lambda \\ &= \sum_{r \in R_y, v \in R_x} \lambda(r)\lambda(v) ryvx[H]_\lambda \end{aligned}$$

Now the set $\{ z[H]_\lambda, z \in G \setminus H \}$ is linearly independent in the group algebra kG. Therefore by comparing the coefficients we get the following .

**PROPOSITION 1.2** $\quad t_{x,y,z} = \sum_{\substack{r \in R_y, v \in R_x \\ ryvx = zh \in zH}} \lambda(r)\lambda(v)\lambda(h)$. □

[4],§67 ) ; no trace in the literature concerning the Hecke algebra $E(G,N,\lambda)$. The aim of this paper is to investigate the structure of this algebra in the case when $G=GL(2,q)$. It turns out that ,although the double cosets of $N$ in $G$ are not as manageable as those of $B$ ( see [4], theorem 65.4) , the generating basis for the Hecke algebra $E(G,N,\lambda)$ satisfy certain natural identities (Proposition 4.9) . We shall prove those identities in §4 after determining a standard basis ( Proposition 3.5 & Theorem 3.6 ) and the structure constants of $E(G,N,\lambda)$ ( Theorem 5.6 ) and use them to characterize the set of idempotents in the Hecke algebra $E(G,N,\lambda)$ (Proposition 6.3). We apply a polynomial algebra technique ( §5 ) to those identities to construct a set of orthogonal idempotents whose sum is the identity of $E(G,N,\lambda)$. Kreig ( [7], Theorem 3.4), proved that any Hecke algebra of dimension $\leq 5$ is commutative. The case we consider here turns out to be commutative and hence provides an example of a commutative Hecke algebra of large dimension. Towards the end of the paper we give a partial characterization of the units in this Hecke algebra.

## §1. HECKE ALGEBRAS AND THEIR STRUCTURE CONSTANTS

Let $G$ be a finite group , $H$ a subgroup of $G$, $k$ is a field and $\lambda : H \to k^{\times}$ be a multiplicative character of $H$. We write $[H]_\lambda = \sum_{h\in H} \lambda(h^{-1})h \in kH$ . It is clear that $[H]_\lambda h = h[H]_\lambda = \lambda(h)[H]_\lambda$, for all $h \in H$ . The left ideal $kG[H]_\lambda$ of kG generated by $[H]_\lambda$, when regarded as a left kG-module, is isomorphic to the induced kG-module $Ind_H^G(L_\lambda)$, where $L_\lambda$ is a one-dimensional $kH$-module which affords $\lambda$. If $x \in G$, write $H^x = x^{-1}Hx$ and let $\lambda^x$ be the multiplicative character of $H^x$ given by $\lambda^x(x^{-1}hx) = \lambda(h)$ for all $h \in H$ . Let $H \backslash G / H = \{ D_x := HxH \ ; x \in I \}$ be the set of distinct $(H,H)$-double cosets of $H$ in $G$ and let $J_\lambda = \{ x \in I \ ; \lambda^x = \lambda \ \ on \ H^x \cap H \}$. Write $Y(H,\lambda) = kG[H]_\lambda$ and let $E(H,\lambda) = End_{kG}(Y(H,\lambda))$; the endomorphism algebra of the kG-module $Y(H,\lambda)$ . If $x \in J_\lambda$, write $H = \bigcup_{r\in R_x} r(H \cap {}^xH)$ and assume that $1 \in R_x$ where 1 is the identity of G. Define $a_x \in E(H,\lambda)$ as follows :

$$a_x([H]_\lambda) = \sum_{r\in R_x} \lambda(r) rx[H]_\lambda$$

# THE STRUCTURE OF A HECKE ALGEBRA FOR THE GENERAL LINEAR GROUPS

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***Abstract:*** *We study the structure of the Hecke algebra $E(N,\lambda)$ where N is the monomial matrices of degree two over a finite field and $\lambda$ is the multiplicative character of N lifted from the sign character of the symmetric group. We give a standard basis and determine the structure constants for this Hecke algebra. We also use this presentation together with a polynomial algebra technique to construct an orthogonal idempotent decomposition in $E(N,\lambda)$ and partially characterize its units. The set of idempotents in this Hecke algebra is also characterized.*

***Keywords:*** *Hecke algebra, Structure constants*

***Mathematical subject classification*** : *Primary 20C33 - Secondary 16S50*

## §0. INTRODUCTION

Hecke algebras play a very important role in the representation of finite groups. One of the most striking examples that show the significance of Hecke algebras in this manner is the Hecke algebra $E(G,B,\mathbf{1_B})$, associated with the triple $(G,B,\mathbf{1_B})$ where $G$ is a finite group of Lie type ,$B$ is a Borel subgroup of $G$ and $\mathbf{1_B}$ is the trivial character of B . Every such group has a structure of split BN-pair $(G,B,N,R,U)$ ( see for example [1] ,[3] ,[5],[6],[10] ). Let $G=GL(n,q)$; the general linear group with coefficient taken from a finite field $\mathbf{F_q}$ ,where $q$ is a power of some prime , with its standard split BN-pair ( see [4] , §65B ) , where $B$ is taken to be the set of upper triangular matrices and N the set of monomial matrices. Then $N$ is an extension of the symmetric group $S_n$ and as such it has a multiplicative character $\lambda$ given by lifting the sign character of $S_n$. Although the Hecke algebra $E(G,B,\mathbf{1_B})$ is a well-studied subject ( see for instance [5] and

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بسم الله الرحمن الرحيم

King Fahd National Library Cataloging - in Publication Data
Khammash, Ahmed A.
The structure of a hecke algebra for the general lilnear group ( GL $(2,p^n)$ - Makkah Al-Mukarramah.
24 P ; 17 X 24 cm
ISBN : 9960 - 03 - 584 -0
1 - Algebra 2 - Algebras, linear I - Title
512 dc 1343 / 23
Legal Deposit no. 1343 / 23
ISBN : 9960-03 - 584 - 0

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# The Structure of A hecke Algebra For The General Linear Group *(GL (2,$p^n$)*

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1424 - 2003